

Universal equations and constants of turbulent motion

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Abstract. This paper presents a parameter-free theory of shear-generated turbulence at asymptotically high Reynolds numbers in incompressible fluids. It is based on a two-fluids concept. Both components are materially identical and inviscid. The first component is an ensemble of quasi-rigid dipole-vortex tubes (vortex filaments, excitations) as quasiparticles in chaotic motion. The second is a superfluid performing evasive motions between the tubes. The local dipole motions follow Helmholtz’ law. The vortex radii scale with the energy-containing length scale. Collisions between quasiparticles lead either to annihilation (likewise rotation, turbulent dissipation) or to scattering (counterrotation, turbulent diffusion). There are analogies with birth and death processes of population dynamics and their master equations and with Landau’s two-fluid theory of liquid Helium. For free homogeneous decay the theory predicts the TKE to follow t^{-1} . With an adiabatic wall condition it predicts the logarithmic law with von Kármán’s constant as $1/\sqrt{2}\pi = 0.399$. Likewise rotating couples form dissipative patches almost at rest (\rightarrow intermittency) wherein under local quasi-steady conditions the spectrum evolves into an “Apollonian gear” as discussed first by Herrmann [1990]. Dissipation happens exclusively at scale zero and at finite scales this system is frictionless and reminds of Prigogine’s (1947) law of minimum (here: zero) entropy production. The theory predicts further the prefactor of the 3D-wavenumber spectrum (a Kolmogorov constant) as $\frac{1}{3}(4\pi)^{2/3} = 1.802$, well within the scatter range of observational, experimental and DNS results.

Keywords: Turbulence, vortex dipoles, vortex tubes, dipole chaos, two-fluid theory, quasiparticles, von Kármán’s constant, law of the wall, Kolmogorov constant

*The diversity of problems in turbulence should not obscure
the fact that the heart of the subject belongs to physics.*

– Falkowski and Sreenivasan (2006).

1. Introduction

Many efforts to solve the turbulence problem rest on the idea that the Navier-Stokes equation (NSE) plays the role of a God equation and the application of a certain number of mathematical operations onto NSE could do it. In particular, the Fridman-Keller [1924] series expansion of NSE played a prominent role, its zeroth element being the Reynolds [1895] equation and

the higher expansion elements subject to various closure hypotheses [for a review see Wilcox, 2006]. This assumption is also reflected in one of the Millenium-Prize problems of the Clay Mathematical Institute announced in 2000. However, until today these efforts could not answer most elementary questions about tur-

bulence. Here we explain why¹.

A methodical alternative was chosen by Prandtl [1926] who discussed turbulence in terms of analogies with molecular diffusion, gas kinetics, and Brownian motion in the interpretation by Einstein [1905]. Prandtl related his mixing length (*Mischungsweg*) with the mean-free path of kinetic gas theory. This concept became popular but detailed questions could not be answered without use of measurements. Although Prandtl has been heavily criticized by Batchelor [1953], also other scientists derived free-hand analogies for turbulence, e.g. the early K - ω model by Kolmogorov [1942], corrected and improved by Saffman [1970], further improved by Wilcox [2006]. They are today part of a larger set of so-called two-equation turbulence models like K - ε , K - $K\mathcal{L}$, K - τ etc.².

Three years before Prandtl, Debye and Hückel [1923] based their theory of electrolytes on the assumption that each ion is surrounded by a spherical "cloud" (or screen) of ions of opposite charge.

The ideas of Prandtl and Debye and Hückel were early examples of constructive theories for large many-particle ensembles in very different branches of physics, where knowledge of parts and pieces does not suffice to describe the system's global behavior.

A similar methodical challenge was met later in the fields of superconductivity and superfluidity by groups around Landau and Feynman [see Landau and Lifschitz, 1983; Landau and Lifshitz, 1987; Feynman, 1955; Anderson, 2011]. They realized that their fundamental equations and principles³ hold perfectly for the single atom, but properties like electric conductivity (or the gloss of metallic surfaces) *emerge* only when a larger *ensemble* of atoms is put together so that new specific features and interrelations come into play, without violating the fundamental equations and principles, of course.

For superfluidity it was necessary to supplement (or even replace) fundamental equations by the two-fluid theory of Landau (1941) – a not too far relative of the ion-cloud concept of Debye and Hückel [1923].

In many cases the formation of ensembles means to break symmetries. So eventually the concept of *emergence* and *broken symmetries* gave physics in the last years a new and much less reductionistic face than it

had until 1923 [Glansdorff and Prigogine, 1971; Haken, 1978; Laughlin, 2005; Anderson, 2011].

Encouraged by the victory of the a. m. two-fluid concept in superfluidity, Liepmann [1961; loc. cit. Spiegel, 1972] and later on Spiegel [1972] made steps to test it for classical fluid turbulence. Along this road, which we follow here too, Spiegel [1972] was the very first who used terms like excitations, quasiparticles and two fluids in the context of classical fluid turbulence, followed later by Spalding [1985]⁴. These authors were at least fully aware of the principle, deep-rooted and almost philosophical problems with reductionistically NSE-based approaches. Ed Spiegel underlined his thorough conviction again just recently in the closing remarks of the report on the final session of the *Turbulence Colloquium Marseilles 2011*⁵. Besides ancient precursors of our ideas like René Descartes (1596 – 1650), who spoke about "tourbillons" forming the universe, and Lord Kelvin, who coined 1867 the notion "vortex atoms" [loc. cit. Saffman, 1992] we have to mention Marmanis [1998] who possibly was the first to propose vortex dipoles as *the* fundamental quasiparticles of turbulence. Finally, the numerical Monte-Carlo eddy-collision methods with various vortex-filament primitives indicate that ideas developed by practitioners are not too far away from our theoretical views [e.g. Andeme, 2008].

Below we elaborate the two-fluid concept in greater detail; not exhaustive because the number of potential applications and side-problems is huge. But we show that, in an idealised sense, turbulence in an incompressible fluid at $Re \rightarrow \infty$ can be understood as a statistical many-body ensemble – a tangle of vortex-dipole tubes (or filaments) taken as interacting *discrete particles*. We will answer a number of open questions of turbulence without use of empirical parameters.

2. Broken symmetry and irreversibility

In his anti-reductionistic article summarizing experiences of condensed-matter physics, Anderson [1972] states that "more is different". The turbulence theory presented below is an extreme example. While the case of *one* quasiparticle (an almost frictionless vortex dipole of zero net circulation) in a volume is non-dissipative and symmetric with respect to time and circulation, already the presence of *two* quasiparticles in a common volume has the potential to break the symmetry and

¹Everywhere the pronomen "we" is used in this text, it means the two of us, the dear reader and the author.

²Comprehensive overviews on the history of turbulence theory can be found in Davidson [2004] and Davidson et al. [2011]

³Schrödinger equation, Pauli principle etc.

⁴see www.archive.org/details/nasa/.techdoc/_19860008217

⁵See <http://turbulence.ens.fr/>

to give rise to the emergence of turbulent dissipation: according to the laws of Helmholtz, dipoles are always in motion, may thus collide and may – depending on the occasional collision angle – form likewise-rotating and thus unstable couples. Their centers of mass stay almost at rest and evolve into a spectrum of smaller and smaller vortices until their kinetic energy is converted into heat at scales of size zero. In a sense we have here a most simple many-body problem because already the transition from $\mathcal{N} = 1$ to $\mathcal{N} = 2$ suffices to break the symmetry and allow the *emergence* of dissipation.

Setting $Re \rightarrow \infty$ in NSE means vanishing viscosity such that only the Euler equation remains. However, the latter has non-unique solutions, visible already in the trivial case of the inviscid Burgers equation. Additional information is needed to achieve uniqueness. I.e. with $Re \rightarrow \infty$ we first delete physics (viscosity) in NSE to be forced then to *add* (from outside) reasonable physics to finally reinstate uniqueness. But this is not all to be overcome. In addition, we have the problem of *localization* and integrability of a solution. Classical weak vortex solutions of the Euler equation extend into the full volume and are thus not integrable. But real-world vortex ensembles exhibit individually localized vortices and finite scales.

Those contradictions could only be resolved introducing quasiparticles embedded in the two-fluid concept initiated earlier by Landau, Tisza, Feynman, Liepmann, Spiegel and Spalding, resting all on Debye and Hückel [1923].

3. Vortex tubes and dipoles: Batchelor couples and von-Kármán couples

Potential vortex. The classical potential vortex around a closed vortex line (e.g. the centerline of a smoke ring) represents an exact weak solution of the Euler equation. The vortex line has infinitely thin diameter, infinitely high angular velocity, but finite circulation, $c = \pi \rho^2 \omega < \infty$, where ρ and ω are radial coordinate and vorticity. The fluid outside the vortex line is inviscid and irrotational and the radial velocity $v = \omega \rho \sim 1/\rho$ decreases with distance ρ from the centerline and extends into space. It is not applicable here, as well as his relative, the Rankine vortex, because they both are filling the space.

Vortex tube. Instead of the above we use the vortex-tube concept, which is a Rankine vortex without sticking condition between its forced and free parts. I.e. its radial velocity increases (like in a rigid vortex) linearly from center to tube radius r , and for $\rho > r$ there

is an inviscid and independent potential flow governed by volume conservation. Thus vorticity is confined to the interior of a spaghetti-like tube as described e. g. by Lugt [1979] and in greater detail by Pullin and Saffman [1998] who quote papers by Kuo & Corrsin, 1972, and Brown & Roshko, 1974, showing tubes as dominating characteristic structures. A more recent simulation study has been presented by Wilczek [2011] in form of a number of instructive movies on vortex ensembles in motion on the internet⁶.

The centerlines of our vortex tubes form either closed loops or they are attached to boundaries. The problem of stability of the tubes is discussed further below.

Batchelor couple. This is a vortex *dipole* made up of two anti-parallel vortices [Lesieur, 1997] and symbolized by $(+ \uparrow -)$ or $(- \uparrow +)$ where plus and minus mean the signs of vorticity within the vortices and the arrow the direction of motion of the couple or dipole. In classical interpretation the flow field of one vortex moves the other vortex and *vice versa*. The total circulation of a dipole is zero because the vorticities carry opposite signs. In practice such couples are stable over moderate propagation times. In our idealized image they are made up of vortex tubes, move frictionless with local center-of-gravity velocity $u = \omega r$ and conserve all their properties excepting their position in space because they propagate with their local u into the same direction as the fluid between the two tubes. In a dense ensemble of Batchelor couples their trajectories are no longer straight lines due to mutual interactions. A couple's kinetic energy density is $u^2/2 = r^2\omega^2/2$.

Von-Kármán couple. It is a counterpart of the Batchelor couple, made up of likewise rotating vortex tubes and symbolized by $(+ \parallel +)$ or $(- \parallel -)$. Its total circulation does not vanish. Such a couple is known since long to be fundamentally unstable. Its kinetic energy is eventually dissipated into heat [e.g. Lamb, 1932]. Further below we discuss details of this process.

4. Dipole chemistry in reaction-diffusion approximation

For 2D trajectories it has been found by Aref [1983] and Eckhardt and Aref [1988] that the trajectories are chaotic so that for very high Re and 3D motions of tube-like vortex dipoles in form of a dense 3D tangle we may assume also chaotic motions where collisions cannot be excluded. Let us consider an asymptotically

⁶<http://pauli.uni-muenster.de/tp/menu/forschen/ag-friedrich/mitarbeiter/wilczek-michael.html>

high tube density so that the chaotic trajectories between collisions are short and locally homogeneous (in a statistical sense), like e. g. in the theory of Brownian motion. Then the total dynamic process of the tangle may be described as [Frisch, 1995] “... conservative dynamics punctuated by dissipative events...”: some “elastic” collisions lead to energy-conserving reorganizations of Batchelor couples, resembling turbulent diffusion, whereas other collisions are dissipative or “inelastic” and lead to the formation of fundamentally unstable von-Kármán couples whose energy moves to smaller and smaller scales where it decays, resembling turbulent dissipation or dipole annihilation.

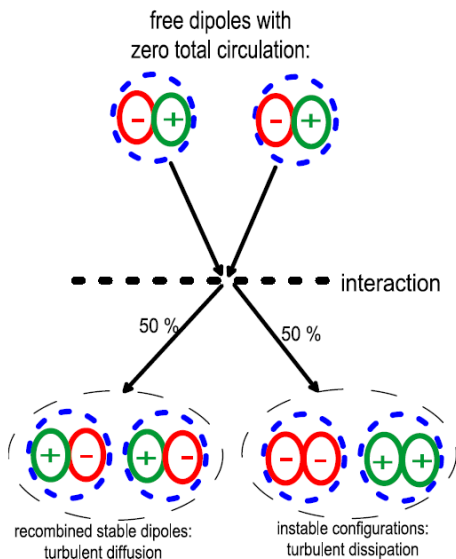


Figure 1. Collisions of two dipoles: the left pathway is “diffusive”; it is a recombination of dipole elements and chaotically scatters the trajectories (turbulent diffusion). The right pathway is “dissipative”; it evolves into an unstable vortex configuration which decays “somehow” into heat.

Figure 1 shows the two potential results of a dipole-dipole collision. For symmetry reasons the two pathways have identical probabilities of $1/2$.

The simplest mathematical structure describing the statistics of the above two irreversible processes of (turbulent) diffusion and (turbulent) dissipation of dipoles is given by the following reaction-diffusion equation, which may be understood also as a special case of the Oregonator [see e.g. Ch. 9 in Haken, 1978]:

$$\frac{\partial \mathcal{N}}{\partial t} + \frac{\partial}{\partial \vec{x}} \left(\vec{U} \mathcal{N} - \nu \frac{\partial \mathcal{N}}{\partial \vec{x}} \right) = \mathcal{F} - \beta \mathcal{N}^2. \quad (1)$$

Here \mathcal{N} is the volume density of dipole tubes (or filaments), ν is turbulent diffusivity, β is a constant, \mathcal{F} is a source term describing the generation of dipole tubes and set equal to zero for the moment. ν and β are unknown so far. We only know for sure that the exponent of \mathcal{N} in the last term of (1) is really 2 because it needs *two* colliding dipoles to generate (with probability $1/2$) unstable configurations which dissipate energy. But if energy is gone, a whole dipole is gone as our dipole tubes as quasiparticles differ from their inviscid potential-flow environment only by their kinetic energy.

The presence of a mean flow and the corresponding advection of dipoles with the flow is sketched for reasons of completeness by the term with mean-flow vector \vec{U} . For simplicity we set further always $\vec{U} \equiv 0$.

One may look at (1) only from the viewpoint of pure analogy with chemical reaction-diffusion processes. But one may also derive this equation over many pages from scratch, beginning with a master equation for the probabilities as sketched earlier [Baumert, 2005b, Ch. 5.6]. Fortunately, this has already been done by other authors many years ago for whole classes of such processes, and went into the many textbooks on stochastic-dynamic systems, physical kinetics and other fields [e.g. Kraichnan, 1968; Haken, 1978, 1983; Stratonovich, 1992, 1994]. Therefore we will not repeat here these formal steps and refer the more technically interested reader to the literature mentioned.

5. Two fluids, dressed and naked tubes

Two fluids. We consider a circulation-free volume filled with two different forms of an materially identical incompressible fluid:

- inviscid, quasi-rigid and incompressible but deformable vortex tubes as quasiparticles with finite radius r and vorticity module ω , vorticity bundled within the tube; in our analogy with the kinetic theory of gases a dipole tube is an analogue of a particle,
- the inviscid fluid *between* the dipole tubes; it is and in the analogous kinetic theory of gases an analog of the vacuum.

Fluid (b) behaves like a super-fluid and receives no force from the moving quasi-rigid dipole tubes (d’Alambert’s paradox). The fluids differ only in their state of motion. While the tubes rotate around their (in general curvilinear) axes and move locally relative to the volume according to Helmholtz’ laws, the fluid between the

tubes performs corresponding evasive motions according to the principle of volume conservation.

Naked tubes. Above we have used the concept of vortex dipoles made up of ideal vortex tubes immersed in an inviscid fluid and exchanging no energy with it. However, this concept is only applicable if the vortex is a rigid body. If it is a fluid, the problem of stability arises because the quasi-rigid vortex tube as an exact solution of the Euler equation is accompanied by the following pressure head as a consequence of inertial (centrifugal) forces:

$$p = p_0 + \frac{\rho}{2} \times \omega^2 r^2. \quad (2)$$

Here p_0 is the background pressure of a laminar reference flow, e.g. in the ocean the depth-depending hydrostatic pressure. If the pressure outside the vortex would simply be p_0 then, due to the action of the outwards-directed pressure force given by the second term in (2), the vortex would loose stability against small perturbations. However, tubes of finite radius are observed to be during their finite life time relatively stable in real-world turbulence.

Dressed tubes. This contradiction can be explained by the consideration of ensemble effects. It has been noted in the Introduction of this paper that in many-body problems like turbulence the phenomenon of *emergence* deserves special attention. This means that in a (local) volume element with a larger number ($\mathcal{N} \gg 1$) of similar vortex tubes (a local “cloud”) the tube ensemble itself generates the background pressure (2) which keeps eventually all the individual tubes – at least in the center of the cloud – in a “sufficiently stable” state.

In thermodynamically open systems like turbulence such a cloud represents a quasi-steady state [*Fließgleichgewicht* in the sense of Bertalanffy, 1953; Glansdorff and Prigogine, 1971; Haken, 1978, 1983]. I.e. the processes of dipole generation (see below) and their annihilation by collisions almost compensate each other. Any quasiparticle (dipole) has thus only a limited statistical life time. Therefore “sufficient stabilization” of a dressed dipole tube by a cloud means to guaranty stability only in a statistical sense during its (statistical) life time.

6. Particle number, TKE, and r.m.s. vorticity frequency

Consider a small volume element δV populated by an ensemble of $j = 1 \dots \mathcal{N}$ dipoles with individual effective vortex radii, r_j , and r.m.s. vorticity moduli, ω_j . The

latter can be interpreted as r.m.s. values of individual dipoles j as follows,

$$\omega_j^2 = \frac{1}{2} [(-\hat{\omega}_j)^2 + (+\hat{\omega}_j)^2] = \hat{\omega}_j^2, \quad (3)$$

where $+\hat{\omega}_j$ and $-\hat{\omega}_j$ are the individual vorticities of the two vortex tubes forming the dipole.

The dipole energy is conserved as long as dissipative events do not take place. The volume density of dipoles is $\mathcal{N}/\delta V$. The total TKE within δV is the sum of the kinetic energies of the individual dipoles:

$$K_{\delta V} = \frac{1}{\delta V} \sum_{j \in \delta V} \frac{1}{2} r_j^2 \omega_j^2 = \frac{\mathcal{N}}{\delta V} \bar{k}, \quad (4)$$

$$\omega_{\delta V}^2 = \frac{1}{\delta V} \sum_{j \in \delta V} \omega_j^2 = \frac{\mathcal{N}}{\delta V} \bar{\omega}^2. \quad (5)$$

Multiplication of (4, 5) with δV gives

$$K = \delta V \cdot K_{\delta V} = \sum_{j \in \delta V} \frac{1}{2} r_j^2 \omega_j^2 = \mathcal{N} \bar{k}, \quad (6)$$

$$\omega^2 = \delta V \cdot \omega_{\delta V}^2 = \sum_{j \in \delta V} \omega_j^2 = \mathcal{N} \bar{\omega}^2. \quad (7)$$

$K_{\delta V}$, $\omega_{\delta V}$ are local volume densities of TKE and r.m.s. vorticity magnitude, respectively. They and K, ω are extensive variables by definition, i.e. they scale with the dipole number in δV . \bar{k} and $\bar{\omega}$ are ensemble averages and as such intensive variables which do not change when new particles with average properties are added to δV :

$$\bar{k} = \frac{1}{\mathcal{N}} \sum_{j \in \delta V} \frac{1}{2} r_j^2 \omega_j^2, \quad (8)$$

$$\bar{\omega}^2 = \frac{1}{\mathcal{N}} \sum_{j \in \delta V} \omega_j^2. \quad (9)$$

We turn now to equation (1) where, according to (6, 7) we replace \mathcal{N} with K/\bar{k} and $\omega/\bar{\omega}$, respectively, to get eventually balance equations for the extensive variables K and $\Omega = \omega/2\pi$, provided that the intensive variables \bar{k} and $\bar{\omega}$ vary sufficiently weakly in time and space compared with the dipole number \mathcal{N} :

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial \vec{x}} \left(\vec{U} K - \nu \frac{\partial K}{\partial \vec{x}} \right) = \mathcal{F}_K - \beta_K K^2, \quad (10)$$

$$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \vec{x}} \left(\vec{U} \Omega - \nu \frac{\partial \Omega}{\partial \vec{x}} \right) = \mathcal{F}_\Omega - \beta_\Omega \Omega^2. \quad (11)$$

Here $\Omega = 1/T$ is the ordinary⁷ vorticity frequency and

⁷While $\omega = 2\pi/T$ is an angular frequency, $\Omega = 1/T$ is an ordinary frequency.

related with ω by the often used constant κ which further below appears to be von-Kármán's constant:

$$\omega = 2\pi\Omega = \Omega/\kappa^2, \quad (12)$$

$$\kappa = (2\pi)^{-1/2} \approx 0.399. \quad (13)$$

We see that

$$\beta_K = \beta/\bar{k}, \quad \beta_\Omega = 2\pi\beta/\bar{\omega}, \quad (14)$$

$$\mathcal{F}_K = \bar{k}\mathcal{F}, \quad \mathcal{F}_\Omega = \bar{\omega}\mathcal{F}/2\pi. \quad (15)$$

7. Mixing length and eddy viscosity

Prandtl [1925] hypothesized that his mixing length has a two-fold meaning. It should be “considered as the diameter of the masses of fluid moving as a whole” (in our picture: as a characteristic vortex radius) or “as the distance traversed by a mass of this type before it becomes blended in with neighboring masses...” (in our picture: as a mean free path). [loc. cit. Bradshaw, 1974]. I.e. he stated a tight relation between the radius and the free path of a dipole in a dense ensemble. Indeed, for a dipole tangle this can be shown. We first define the ensemble average, \bar{r} , of the vortex radii as an average over a volume element weighted by vorticity as follows,

$$\bar{r}^2 = \frac{\sum r_j^2 \omega_j^2}{\sum \omega_j^2} = \frac{2K}{\omega^2} = \frac{2K}{(2\pi\Omega)^2}. \quad (16)$$

We now determine the eddy viscosity, ν , in analogy to the theory of Brownian motion [Einstein, 1905] in terms of a mean free path, λ , and a mean free flight time, $\tau = \lambda/\bar{u}$:

$$\nu = \lambda^2/2\tau. \quad (17)$$

Here \bar{u} is the mean velocity of a dipole,

$$\bar{u}^2 = 2K = \sum u_j^2 = \sum r_j^2 \omega_j^2. \quad (18)$$

For a turbulent tangle of dipole tubes at $Re = \infty$ in a quasi-steady state far from solid boundaries the mean free path will clearly be short but cannot vanish because there is an equilibrium of dipole generation, annihilation, and motions that lead to the latter. This implies that [Baumert, 2005b]

$$\lambda = 2\bar{r}. \quad (19)$$

Algebra finally gives the following:

$$\nu = K/\pi\Omega. \quad (20)$$

This relation is called the Prandtl-Kolmogorov relation.

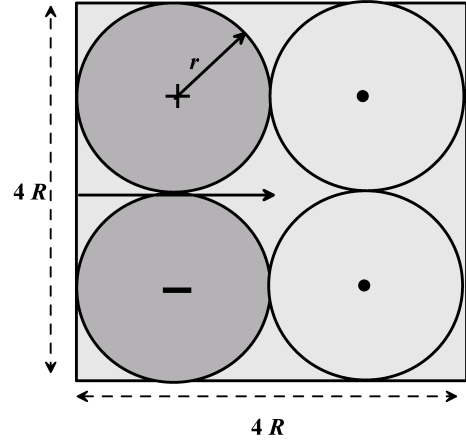


Figure 2. Local cross section through a dipole tangle far from boundaries at maximum dipole density ($Re \rightarrow \infty$). The dark grey dipole in the left half square labels the initial situation, the right half square a situation after a translation motion from left to right into its collision position, where it is either annihilated or scattered. A free dipole obviously occupies statistically a cross sectional area of $(4\bar{r})^2$.

8. Parameters β_K , β_Ω , and ζ

We consider an initial-value problem ($\mathcal{F}_K = 0$, $\mathcal{F}_\Omega = 0$) in a spatially homogeneous volume where $\partial K/\partial \vec{x} = 0$ and $\partial \Omega/\partial \vec{x} = 0$ such that equations (10, 11) reduce to

$$\frac{dK}{dt} + \beta_K K^2 = 0, \quad (21)$$

$$\frac{d\Omega}{dt} + \beta_\Omega \Omega^2 = 0. \quad (22)$$

For large t follows that

$$K(t) = (\beta_K t)^{-1}, \quad (23)$$

$$\Omega(t) = (\beta_\Omega t)^{-1}, \quad (24)$$

and, according to (20), for eddy viscosity holds

$$\nu(t) = \frac{K(t)}{\pi\Omega(t)} = \frac{\beta_\Omega}{\pi\beta_K} = \text{const.} \quad (25)$$

Equation (23) coincides with the results of a fairly general similarity analyses of the Navier-Stokes equations by Oberlack [2002] and with experimental results by Dickey and Mellor [1980].

We now use the definition of the dissipation rate, ε , which is the last term on the right-hand side of (10) and carries the units of of TKE (density) per time, i.e. $\varepsilon \sim K/T \sim K\Omega$, [$\text{m}^2 \text{s}^{-3}$]. For reasons of convenience

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we write this variable as follows and introduce a still unknown auxiliary variable, ζ :

$$\varepsilon = \beta_K K^2 = \zeta \frac{K \Omega}{\pi} = \zeta \frac{K^2}{\nu \pi^2}, \quad (26)$$

where we made use of (20). This implies

$$\beta_K = \zeta / \nu \pi^2. \quad (27)$$

When the dipoles behave in general like in free decay then we may use (25) which gives then the expressions

$$\beta_\Omega = \zeta / \pi, \quad (28)$$

$$\beta_K = \zeta / \nu \pi^2. \quad (29)$$

With these results we may rewrite (10, 11) as follows:

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial \vec{x}} \left(\vec{U} K - \nu \frac{\partial K}{\partial \vec{x}} \right) = \mathcal{F}_K - \frac{\zeta K^2}{\nu \pi^2}, \quad (30)$$

$$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \vec{x}} \left(\vec{U} \Omega - \nu \frac{\partial \Omega}{\partial \vec{x}} \right) = \mathcal{F}_\Omega - \frac{\zeta}{\pi} \Omega^2, \quad (31)$$

with ν given by (20) and $\zeta \equiv 1$ as shown further below.

9. TKE and vorticity generation by mean-flow shear

Generation of TKE. To further complete equations (30, 31), we have, besides ζ , to specify the source terms $\mathcal{F}_K, \mathcal{F}_\Omega$. The specification of \mathcal{F}_K for shear flows is trivial because it is given by the classical losses of the mean-flow energy balance and can therefore be copied from textbooks like Wilcox [2006] or Schlichting and Gersten [2000]:

$$\begin{aligned} \mathcal{F}_K &= - \sum_{i,j=1}^3 \langle u'_i u'_j \rangle \frac{\partial U_i}{\partial x_j} \\ &= \nu S^2 - \frac{2}{3} K \sum_{i,j=1}^3 \delta_{ij} \frac{\partial U_i}{\partial x_j}, \end{aligned} \quad (32)$$

where $-\langle u'_i u'_j \rangle$ is the Reynolds stress tensor defined as

$$-\langle u'_i u'_j \rangle = 2 \nu S_{ij} - \frac{2}{3} \delta_{ij} \cdot K \quad (33)$$

and S_{ij} is the rate of strain tensor defined as

$$S_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right). \quad (34)$$

S^2 is the total instantaneous shear squared,

$$S^2 = \sum_{i,j=1}^3 \left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right) \frac{\partial U_i}{\partial x_j}, \quad (35)$$

U_i is the i^{th} component of the mean flow velocity vector $\vec{U} = (U_1, U_2, U_3)^T$ and δ_{ij} is the Kronecker symbol which is zero for $i \neq j$ and unity for $i = j$. In the simple case of a plane horizontal flow with vertical shear like wind over plane terrain or flow in a plane wide channel we have $i = 1$ and $j = 3$ such that (32) reads as follows:

$$\mathcal{F}_K = \nu S^2. \quad (36)$$

Generation of vorticity. We use a fundamental macroscopic argument by Tennekes [1989]. It has first been cast into mathematical form by Baumert and Peters [2000, 2004] and carefully discussed by Kantha [2004, 2005], by Kantha et al. [2005] and by Kantha and Clayson [2007]. Tennekes hypothesized that, in a neutrally stratified homogeneous shear flow, an energy-containing turbulent length scale, \mathcal{L} , cannot, on dimensional grounds, depend on the ambient shear.

Further, also on dimensional grounds we have $L \sim K^{1/2} \Omega^{-1}$ and thus $L^2 \sim K \Omega^{-2}$ such that the time evolution of the length scale follows

$$\frac{1}{L^2} \frac{dL^2}{dt} \sim \frac{1}{K} \frac{dK}{dt} - 2 \frac{1}{\Omega} \frac{d\Omega}{dt}. \quad (37)$$

We replace dK/dt with \mathcal{F}_K according to (32) and find

$$\frac{1}{L^2} \frac{dL^2}{dt} \sim \frac{S^2}{\pi \Omega} - 2 \frac{1}{\Omega} \frac{d\Omega}{dt}. \quad (38)$$

Tennekes' hypothesis means that the evolution of L cannot be controlled by S which means $dL^2/dt = 0$ and implies that

$$\frac{S^2}{\pi \Omega} = \frac{2 \mathcal{F}_\Omega}{\Omega}. \quad (39)$$

where we replaced $d\Omega/dt$ with \mathcal{F}_Ω . Algebra gives

$$\mathcal{F}_\Omega = S^2 / 2 \pi \quad (40)$$

and for the simple case of a plane horizontal flow with vertical shear we may complete with some algebra equations (30, 31) as follows:

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial \vec{x}} \left(\vec{U} K - \nu \frac{\partial K}{\partial \vec{x}} \right) = \nu (S^2 - \zeta \Omega^2), \quad (41)$$

$$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \vec{x}} \left(\vec{U} \Omega - \nu \frac{\partial \Omega}{\partial \vec{x}} \right) = \frac{1}{\pi} \left(\frac{S^2}{2} - \zeta \Omega^2 \right) \quad (42)$$

with ν again given by (20).

10. Turbulent boundary layer

Boundary conditions. Consider a horizontally homogeneous flow and its stationary boundary layer close to

a plane solid wall at $x_3 = z = 0$ where for convenience z is introduced here as the only relevant coordinate, pointing orthogonal from the wall into the fluid. Thus the shear is

$$S = |dU/dz|. \quad (43)$$

This special case means that in (73, 74) the horizontal and time derivatives vanish and it remains

$$-\frac{d}{dz} \left(\nu \frac{dK}{dz} \right) = \nu (S^2 - \zeta \Omega^2), \quad (44)$$

$$-\frac{d}{dz} \left(\nu \frac{d\Omega}{dz} \right) = \frac{1}{\pi} \left(\frac{S^2}{2} - \zeta \Omega^2 \right) \quad (45)$$

The diffusive TKE flux into the viscous sublayer at $z = z_0$ has to vanish in the sense of an adiabatic boundary condition,

$$(\nu dK/dz)_{z=z_0} = 0, \quad (46)$$

such that also the flux divergence on the left-hand side of (44) vanishes, giving $K = K_0$ and $\nu (S^2 - \zeta \Omega^2) = 0$ or

$$\Omega = S/\sqrt{\zeta}. \quad (47)$$

Logarithmic law of the wall. We insert (47) into (45). The unknown ζ cancels out⁸ and we have to solve the following equation for $S = S(z)$,

$$2K_0 \frac{d}{dz} \left(\frac{1}{S} \frac{dS}{dz} \right) = S^2, \quad (48)$$

which gives

$$S(z) = \frac{dU}{dz} = \frac{\sqrt{2K_0}}{z}. \quad (49)$$

Integration of (49) gives the logarithmic law of the wall.

In boundary layer theory the bottom shear stress is defined in terms of the squared friction velocity, u_f^2 ,

$$u_f^2 = \nu \frac{dU}{dz} = \frac{K_0}{\pi \Omega} S, \quad (50)$$

and with (47) it follows that

$$K_0 = \pi u_f^2 / \sqrt{\zeta}. \quad (51)$$

This allows to rewrite (49) as follows,

$$\frac{dU}{dz} = \frac{u_f}{\tilde{\kappa} z}, \quad (52)$$

with $\tilde{\kappa}$ as a modified von-Kármán constant defined with respect to (15) through

$$\tilde{\kappa} = \kappa \zeta^{1/4}. \quad (53)$$

Integration of (52) provides us with

$$U(z) = \frac{u_f}{\tilde{\kappa}} \ln \left(\frac{z}{z_0} \right). \quad (54)$$

Mixing length L . Consider the definition of the effective (statistically averaged) dipole radius of an ensemble through (16). We solve this equation for K and express the TKE in terms of \bar{r} and Ω as follows:

$$K = 2\pi^2 \bar{r}^2 \Omega^2. \quad (55)$$

Following now Hinze [1959, p. 279, eq. 5-2], in present notation Prandtl's mixing length L , which is termed also the “energy-containing length scale” in the literature, is defined in terms of eddy viscosity and shear as follows:

$$\nu = L^2 \left| \frac{dU}{dz} \right| = L^2 S. \quad (56)$$

Due to the eddy-viscosity formula (20) relation (56) gives

$$L^2 = \frac{K}{\pi \Omega^2 \zeta^{1/2}}, \quad (57)$$

so that in the neighborhood of a solid wall we get with (47) the following result:

$$K = \pi L^2 \Omega^2 \sqrt{\zeta}. \quad (58)$$

Comparing (55) with (58) gives

$$L = \bar{r}/\tilde{\kappa}. \quad (59)$$

The physical meaning of L can be understood as follows (see Fig. 3). If we may set $\zeta = 1$ then (59) gives together with (15) and (53) the following relation:

$$L^2 = 2 \times (\pi \bar{r}^2). \quad (60)$$

It means that L is the length of a square with an area equal to the cross sectional area of a dipole (in a statistically averaged sense) because $\pi \bar{r}^2$ is the cross-sectional area of one vortex tube. In an asymptotic sense this case corresponds to the maximum deformation of a dipole and justifies to set $\zeta \equiv 1$.

While Prandtl's mixing-length concept was applicable only in the vicinity of solid boundaries so that it attracted respectful criticism Wilcox [2006] our concept is a generalization of Prandtl's concept and works also far

⁸The respective text in <http://arxiv.org/pdf/0907.0223.pdf> contains an algebraic error but without later consequences, fortunately.

from boundaries, even in the free stream of stratified fluids where L may approach the Thorpe scale and/or the Ozmidov scale, depending on the conditions (Baumert and Peters 2004).

We summarize this section as follows:

$$\nu = u_f L, \quad (61)$$

$$L = \kappa z, \quad (62)$$

$$z = L/\kappa = \bar{r}/\kappa^2. \quad (63)$$

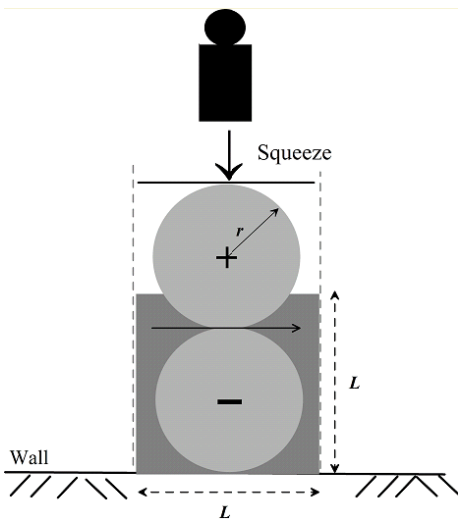


Figure 3. Cross section through a dipole sheet at a solid boundary. In this Figure we have $r = \bar{r}$, i.e. the sketched quasiparticle is to be understood as an ensemble average.

11. Spectra, dissipative patches, and spectral constants

Spectra. Up to now our discussion was concentrated on only a few scales like \bar{r} and L interrelated by κ , and on T interrelated with Ω and ω etc. However, the reality of turbulence exhibits a whole range of scales at which fluctuations occur. They have relatively stable spectral properties. This problem has attracted early attention by Richardson [1922] and Taylor [1937]. Kolmogorov [1941] found on dimensional grounds that the kinetic energy spectrum as function of wavenumber, wherein energy flows steadily from the large (energy-containing) scales to the much smaller dissipative scales, may be written as follows,

$$dK = \alpha_1 \varepsilon^{\alpha_2} k^{-\alpha_3} dk. \quad (64)$$

Here $k = 2\pi/\Lambda$ is the wave number and Λ the wavelength. ε is the dissipation rate of TKE. Dimensional arguments force that that $\alpha_2 = 2/3$ and $\alpha_3 = 5/3$, in agreement with the oceanographic observations by Grant et al. [1959] in a tidal inlet with $Re \approx 10^8$ and a depth of about 100 m. A value for α_1 is derived below.

Devil’s gear. Our view of the above Kolmogorov-Richardson cascade has been filled with life through a numerical simulation study by Herrmann [1990] who demonstrated that Kolmogorov’s value for α_3 corresponds numerically to the data of a space-filling bearing [see also Herrmann et al., 1990]. The latter is the densest non-overlapping (Apollonian) circle packing in the plane, with side condition that the circles are pointwise in contact but able to rotate freely, without friction or slipping [a “devil’s gear” *sensu* Pöppe, 2004]. The contact condition for two different “wheels” with indices 1 and 2 of the gear reads

$$u = \omega_1 r_1 = \omega_2 r_2, \quad (65)$$

where u is necessarily constant throughout the gear and governed by the energy of the decaying (initially energy-containing) vortex pair as $u = \sqrt{2K}$. It follows that

$$\omega_2 = \omega_1 \frac{r_1}{r_2}, \quad (66)$$

and for very small r_2 the frequency ω_2 may become high, even acoustically relevant.

The dissipative patch. If the above gear is frictionless then the question arises where energy can be dissipated. In a real fluid with non-zero viscosity, dissipation happens at all scales, mainly but not exclusively where the velocity gradient is highest, here: at a scale vanishing with $Re \rightarrow \infty$ to the size zero. Our dissipative patch (Fig. 4 shows the first stage of its formation) is thus “almost frictionless” and a Hamiltonian clockwork, excepting scales of size zero.

The formation of a fully developed spectrum of “wheels” from Fig. 4 deserves certain perturbations “from the sides”, a condition which is guaranteed by the random reconnection/recombination and scatter processes sketched in the left half of Fig. 1 and also by the incomplete mutual pressure compensation of the vortices in our vortex ensemble. In a quasi-steady state these perturbations initiate roll-up instabilities at the boundaries of the respective larger vortices so that eventually and in a statistical sense a patch like in Fig. 4 is formed and evolves steadily into a fully developed gear.

Our Fig. 1 illustrates the possible results of a dipole-dipole collision. While the left pathway shows the recombination of *counter*-rotating vortices from counter-

rotating vortices, the right shows the formation of a couple of *likewise* rotating vortices from counter-rotating vortices. The latter then revolve around a common center of mass which remains nearly at rest (Fig. 4). Such a couple is fundamentally unstable [Sommerfeld, 1948], a quasi-steady dissipative patch evolving into a full gear in the sense of the mechanisms discussed in the last Chapter. This picture lets us expect that dissipation should exhibit a spatially *patchy* behaviour which we may also call *intermittency*. This problem has been studied extensively by various authors from other points of view [see e.g. Frisch, 1995] and cannot yet be discussed here from our viewpoint in greater detail.

Kolmogorov's constant α_1 . This constant belongs to the wavenumber spectrum and deserves an idea about the outer limits and inner structure of an unstable, dissipative patch as sketched in Fig. 5 for the begin of the cascade process evolving into a structure like the one given in Fig. 4.

The most important message of Fig. 5 is that the *longest* or energy-containing wavelength of the dissipative patch equals $\Lambda_0 = 2\bar{r}$. The wavelength in a dipole is $4\bar{r}$. The dipole performs chaotic trajectories in a white-noise sense and forms no patch or spectrum. This difference between the two configurations is essential. We use our Λ_0 as a lower integration limit for the spectral energy distribution. It is important to underline that Λ_0 labels the upper wavelength limit (the longest wavelike motions) in a dissipative patch. This limit is actually not influenced by the formation details of the spectrum.

We integrate (64) over the dissipative patch in the sense sketched in Figures 4 and 5 and get

$$K = \alpha_1 \varepsilon^{2/3} \int_{k_0}^{\infty} k^{-5/3} dk = \alpha_1 \frac{3}{2} \left(\frac{\varepsilon}{k_0} \right)^{2/3}, \quad (67)$$

where $k_0 = 2\pi/\Lambda_0$ characterizes the lower end of the turbulence spectrum in the wavenumber space. We loosely assign the wavenumber range $k = 0 \dots k_0$ to the mean flow which may basically be resolved in numerical models. The dissipation rate ε in (67) can be expressed as follows,

$$\varepsilon = K/\tau, \quad (68)$$

with τ being the lifetime of a dissipative patch. Inserting (68) in (67) and rearranging gives the following:

$$\alpha_1 = \frac{2}{3} (2\pi)^{2/3} K^{1/3} \left(\frac{\tau}{2\bar{r}} \right)^{2/3}. \quad (69)$$

In a local quasi-equilibrium sense for a dense vortex ensemble the marching dipoles can occupy only those

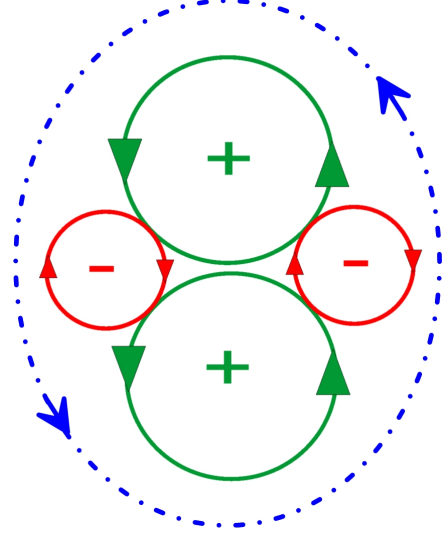


Figure 4. Cross section through the first developmental stage of a dissipative patch, i.e. of an unstable pair of likewise rotating vortices resulting from a dipole-dipole collision (right pathway in Fig. 1). The green circles represent primary energy-containing vortices. They do *not* touch each other due to spontaneously formed secondary vortices which initiate a whole vortex cascade. The broken blue line and the arrows symbolize the slow rotation of the patch around its center of mass.

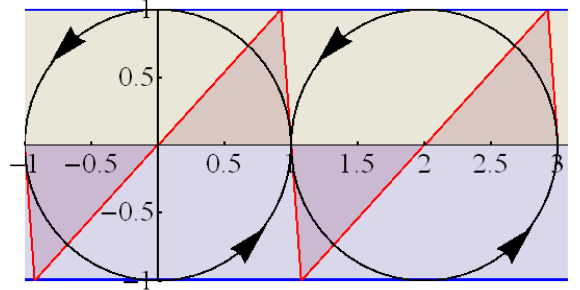


Figure 5. Outer limits of a dissipative patch (*c.f.* Fig. 4). The maximum wavelength is obviously equal to $\Lambda_0 = 2\bar{r}$.

places which are simultaneously also “emptied” through dipole annihilation or dissipative patches by decay. This means that the life time of a dissipative patch, $\tau = K/\varepsilon$, should equal the time of “free flight” of a dipole over a distance $2\bar{r}$:

$$\tau = K/\varepsilon = 2\bar{r}/u. \quad (70)$$

Here we used the scalar dipole velocity u ,

$$u = \omega r = \sqrt{2K}. \quad (71)$$

After some algebra we get the pre-factor of the three-dimensional wavenumber spectrum as follows:

$$\alpha_1 = \frac{1}{3}(4\pi)^{2/3} = 1.802. \quad (72)$$

The corresponding value of an ideal one-dimensional spectrum is one third of the above, i.e. 0.60.

12. Discussion

Equations of turbulent motion. The results of our considerations can be summarized as follows:

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial \vec{x}} \left(\vec{U} K - \nu \frac{\partial K}{\partial \vec{x}} \right) = \nu (S^2 - \Omega^2), \quad (73)$$

$$\frac{\partial \Omega}{\partial t} + \frac{\partial}{\partial \vec{x}} \left(\vec{U} \Omega - \nu \frac{\partial \Omega}{\partial \vec{x}} \right) = \frac{1}{\pi} \left(\frac{S^2}{2} - \Omega^2 \right), \quad (74)$$

$$\nu = \frac{K}{\pi \Omega}. \quad (75)$$

These equations are structurally identical with the k - ω closure model discussed by Wilcox [2006]. There are only slight differences in the pre-factors of the terms.

This theory applies exclusively to locally homogeneous, isotropic and moderately unsteady high-Reynolds number flows. Extreme non-stationarities and/or sharp spatial gradients like in shockwaves are possibly not covered. As a rule, temporal changes of the mean flow should happen on time scales sufficiently long compared with $T = 1/\Omega$ because otherwise spectral universality (64) has possibly not enough time to become well enough established.

Our equations reproduce the logarithmic law of the wall⁹ and predict the universal von-Kármán's constant as $\kappa = 1/\sqrt{2\pi} = 0.399 \approx 0.4$ where the latter value counts as internationally accepted standard [see also Höglström, 1985]. The value corresponds nicely to measurements under *favorable* pressure gradients [Chauhan et al., 2005]. Similar support comes from Jimenez and Moser [2007] on the basis of an extensive review. They state: *The Kármán constant $\kappa \approx 0.4$ is approximately universal.*

However, Landau and Lifshitz [1987] wrote on their p. 173 that *... κ (is) a numerical constant, the von-Kármán constant, whose value cannot be calculated theoretically and must be determined experimentally. It is found to be $\kappa = 0.4$.*

⁹The controversy of the logarithmic *versus* a power-law boundary layer [Barenblatt, 1997] shall not be discussed here.

With respect to the measurability of κ even the opposite might be true: if “physics disappears” when $Re \rightarrow \infty$ and only sort of “inert geometry” (the Euler equation) remains, κ can possibly no longer be seen as a physical quantity as it represents pure geometry.

But these interesting speculations shall not rise any doubt about the necessity of superpipe facilities for very high- Re number experiments like in Princeton [Zagarola and Smits, 1998], Oregon [Donnelly, 1998] or in the European CICLoPE [Talamelli et al., 2009].

Our equations describe the free decay of turbulence following $K \sim t^{-m}$ with $m = 1$, in agreement with Dickey and Mellor's [1980] high- Re laboratory experiments and with Oberlack's [2002] theoretical result for the Navier-Stokes equation. Today it is still not clear why some decay experiments lead to $m > 1$. Possibly it is a matter of initial conditions [see Hurst and Vassilicos, 2007]: at high Re viscosity is comparatively small so that its regularizing effect towards a finally more self-similar decay spectrum will take more time than at lower Re . In some cases this time may exceed the lifetime of turbulence. This underlines the necessity of deeper experimental work.

Kolmogorov's constants. The rounded numerical values $\alpha_1 = 1.8$ or $\alpha_1/3 = 0.6$ predicted by our theory for Kolmogorov's universal constant are situated well within the error bars of many high- Re observations, NSE and RG based analytical approximations, laboratory and DNS experiments.

Based on observations, Tennekes and Lumley [1972] gave the value $\alpha_1 = 1.62$, but still with some uncertainty. The study by Sreenivasan [1995] (see Fig. 6) is possibly the most comprehensive review of experimental and observational values for the number $\alpha_1/3$ until now. Later Yeung and Zhou [1997] reported a value of $\alpha_1 = 1.62$ based on high-resolution DNS studies with up to 512^3 grid points. A most recent study by Donzis and Sreenivasan [2010] on a DNS grid of 4096^3 gave $\alpha_1 \approx 1.58$.

Much higher Reynolds numbers than in DNS could be achieved in oceanic measurements of Lagrangian frequency spectra by Lien and D'Asaro [2002]. These authors stated for the prefactor β_1 in the Lagrangian frequency spectrum that *... since the present uncertainty is comparable to that between high quality estimates of the Eulerian one-dimensional longitudinal Kolmogorov constant measured by many dozen investigators over the last 50 years, large improvements in the accuracy of the estimate of β_1 seem unlikely.* For completeness reasons we mention other theoretical efforts to calculate the universal constants. They are technically extremely com-

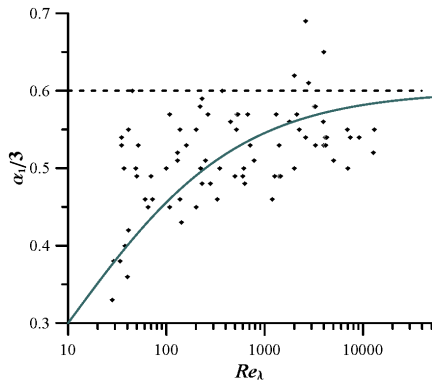


Figure 6. Experimental and observational results for $\alpha_1/3$ measured, collected from the literature, and analyzed by Sreenivasan [1995]. The solid line follows our somewhat arbitrary approximation $0.6 \times \sqrt{Re_\lambda} / (\sqrt{Re_*} + \sqrt{Re_\lambda})$ wherein $\alpha_1/3 = 0.6$ is our theoretically derived asymptotic value. Here we took $Re_* = 10$.

plex and neither unique nor part of an integrated descriptive concept for turbulence so that they all carry more singular characters. Beginning with an initiating work by Forster et al. [1977], systematic analytical approximations using RG methods and related techniques for NSE gave rise to some estimates. E.g. Yakhot and Orszag [1986b, a] found $\alpha_1 \approx 1.62$ whereas McComb and Watt [1992] derived $\alpha_1 = 1.60 \pm 0.01$ and Park and Deem [2003] obtained $\alpha_1 = 1.68$.

Coda. Saffman [1977] [loc. cit. Davidson, 2004, p. 107] feared that ... *in searching for a theory of turbulence, perhaps we are looking for a chimera.*

This has recently been enforced by Hunt [2011]: ... *But there are good reasons why the answer to the big question that Landau and Batchelor raised about whether there is a general theory of turbulence is probably ‘no’.*

Due to our new theory which draws some profit namely from the two-fluid ideas of Landau we are more optimistic. But skeptical thoughts remain in view of the extraordinary low measurement accuracy of the universal constants of turbulence compared with the extremely high precision of the fundamental constants of physics like e.g. the vacuum speed of light, or the mass of the proton [Fritzsche, 2009]. The challenge is possibly the dynamic character of turbulence which is in the best case characterized as a stable steady state rather than a static property like the electron’s electric charge.

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